A Note on Darboux Polynomials of Monomial Derivations☆

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Abstract

We study a monomial derivation d proposed by J. Moulin Ollagnier and A. Nowicki in the polynomial ring of four variables, and prove that d has no Darboux polynomials if and only if d has a trivial field of constants.

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1. Introduction

Throughout this paper, let $k[X] = k[x_1, x_2, \ldots, x_n]$ denote the polynomial ring over a field k of characteristic 0.

A derivation $d = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}$ of $k[X]$ is said to be a monomial derivation if each f_i is a monomial in $k[X]$. By a Darboux polynomial of d we mean a polynomial $F \in k[X]$ such that $F \notin k$ and $d(F) = \Lambda F$ for some $\Lambda \in k[X]$.

Derivations and Darboux polynomials are useful algebraic methods to study polynomial or rational differential systems. If we associate a polynomial differential system $\frac{d}{dt}x_i = f_i$, $i = 1, \ldots, n$, with a derivation $d = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}$, then the existence of Darboux polynomials for d is a necessary condition for the system to have a first integral (see [1, 2, 3]). Darboux polynomials also have important applications in many branches of mathematics. The famous Jacobian conjecture for $k[X]$ is equivalent to the assertion that $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ is, apart from a polynomial coordinate change,

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the only commutative $k[X]$-basis of $\mathrm{Der}_k\ k[X]$. It is proved that n pairwise commuting derivations form a commutative basis if and only if they are k-linearly independent and have no common Darboux polynomials [4].

The most famous derivation without Darboux polynomials may be the Jouanolou derivation, there are several different proofs on the fact that Jouanolou derivations have no Darboux polynomials, see[5, 6]. It is obvious that if d is without Darboux polynomials, then the field $k(X)^d$ is trivial. The opposite implication is, in general, not true. In [7], there is a full description of all monomial derivations of $k[x, y, z]$ with trivial field of constants. Using this description and several additional facts, Moulin-Ollagnier and Nowicki present full lists of homogeneous monomial derivations of degrees $s \leq 4$ (of $k[x, y, z]$) without Darboux polynomials in [8] and then in [9], they prove that a monomial derivation d (of $k[x, y, z]$) has no Darboux polynomials if and only if d has a trivial field of constants and $x_i \nmid d(x_i)$ for all $i = 1, \ldots, n$.

More precisely, look at a monomial derivation d of $k[X]$ with $d(x_i) = x_1^{\beta_{i1}} \cdots x_n^{\beta_{in}}$ for $i = 1, \ldots, n$ and each β_{ij} is a non-negative integer. In this case, d is said to be normal monomial if $\beta_{11} = \beta_{22} = \cdots = \beta_{nn} = 0$ and $w_d \neq 0$, where w_d is the determinant of the matrix $[\beta_{ij}] - I$. In [9], it is proved that if d is a normal monomial derivation of $k[X]$, then d is without Darboux polynomials if and only if $k(X)^d = k$. What happens if $w_d = 0$? In [9], a monomial derivation d of $k[x, y, z, t]$ with $w_d = 0$ defined by

$$d(x) = t^2, d(y) = zt, d(z) = y^2, d(t) = xy$$

is proposed. In this note, we prove that d has no Darboux polynomial if and only if d has a trivial field of constant.

2. Main Results

Now we recall some lemmas related to Darboux polynomials of polynomial derivations. Denote by $A_\gamma^{(s)}$ the group of all γ-homogeneous polynomials of degree s in $k[X]$. Then $k[X]$ becomes a γ-graded ring $k[X] = \oplus_{s\in\mathbb{Z}} A_\gamma^{(s)}$. Recall that D is said to be a γ-homogeneous derivation of degree s if $D(A_\gamma^{(p)}) \subseteq A_\gamma^{(s+p)}$ for any $p \in \mathbb{Z}$.

Lemma 2.1. [10, Proposition 2.2.1] *Let f be a Darboux polynomial of D. Then all factors of f are also Darboux polynomials of D.*

Lemma 2.2. [10, Proposition 2.2.3] *Let D be a γ-homogeneous derivation of degree s and f be a Darboux polynomial of D and λ be a polynomial eigenvalue of f with respect to D. Then λ is a γ-homogeneous polynomial of degree s, and every γ-homogeneous component of f is also a Darboux polynomial of D with polynomial eigenvalue λ.*

Now consider a monomial derivation d defined by $d(x_i) = X^{\beta_i}$, where $\beta_i = (\beta_{i1}, \ldots, \beta_{in}) \in \mathbb{N}^n$. Write $\beta = [\beta_{ij}]$, $\alpha = [\alpha_{ij}] = \beta - I$, where I is the identity matrix of order n. Let $w_d = \det \alpha$, that is,

$$w_D = \det \alpha = \begin{vmatrix} \beta_{11} - 1 & \beta_{12} & \ldots & \beta_{1n} \\ \beta_{21} & \beta_{22} - 1 & \ldots & \beta_{2n} \\ \vdots & \vdots & & \vdots \\ \beta_{n1} & \beta_{n2} & \ldots & \beta_{nn} - 1 \end{vmatrix}.$$

Look at the monomial derivation d of $k[x, y, z, t]$ defined by

$$d(x) = t^2, d(y) = zt, d(z) = y^2, d(t) = xy.$$

Theorem 2.3. *d has no Darboux polynomials if and only if d has a trivial field of constants.*

Proof. It is obvious that if d is without Darboux polynomials, then the field $k(X)^d$ is trivial.

Now suppose that $k(X)^d$ is trivial. Assume that d has a Darboux F such that $d(F) = \Lambda F$. Since d is a homogeneous derivation of degree 1, then by Lemma 2.2, we have Λ is a homogeneous polynomial of degree 1, thus $\Lambda = k_1x + k_2y + k_3z + k_4t, k_1, \ldots, k_4 \in k$.

Let $\sigma : k[x, y, z, t] \to k[x, y, z, t]$ be an automorphism defined by:

$$\sigma(x) = \varepsilon^3 x, \quad \sigma(y) = \varepsilon^5 y, \quad \sigma(z) = \varepsilon^3 z, \sigma(t) = \varepsilon t,$$

where ε is a primitive eighth root of 1. Then σ^{-1} is:

$$\sigma^{-1}(x) = \varepsilon^5 x, \quad \sigma^{-1}(y) = \varepsilon^3 y, \quad \sigma(z)^{-1} = \varepsilon^5 z, \sigma^{-1}(t) = \varepsilon^7 t.$$

It is easy to verify that

$$\sigma^{-1} d\sigma(x) = \sigma^{-1} d(\varepsilon^3 x) = \sigma^{-1}(\varepsilon^3 t^2) = \varepsilon^{17} t^2 = \varepsilon t^2,$$

$$\sigma^{-1} d\sigma(y) = \sigma^{-1} d(\varepsilon^5 y) = \sigma^{-1}(\varepsilon^5 zt) = \varepsilon^{17} zt = \varepsilon zt,$$

$$\sigma^{-1}d\sigma(z) = \sigma^{-1}d(\varepsilon^3 z) = \sigma^{-1}(\varepsilon^3 y^2) = \varepsilon^9 y^2 = \varepsilon y^2,$$
$$\sigma^{-1}d\sigma(t) = \sigma^{-1}d(\varepsilon t) = \sigma^{-1}(\varepsilon xy) = \varepsilon^9 xy = \varepsilon xy.$$

Thus,

$$\sigma^{-1}d\sigma = \varepsilon d, \text{moreover}, \;\; \sigma^{-i}d\sigma^i = \varepsilon^i d.$$

Let

$$\bar{F} = \prod_{i=0}^{7} \sigma^i(F), \quad \bar{\Lambda} = \sum_{i=0}^{7} \varepsilon^i \sigma^i(\Lambda).$$

Then

$$\begin{aligned} d(\bar{F}) &= d(\prod_{i=0}^{7} \sigma^i(F)) = \sum_{i=0}^{7} \sigma^0(F) \cdots d(\sigma^i(F)) \cdots \sigma^7(F) \\ &= \sum_{i=0}^{7} \sigma^0(F) \cdots \varepsilon^i \sigma^i(d(F)) \cdots \sigma^7(F) \\ &= \sum_{i=0}^{7} \sigma^0(F) \cdots \varepsilon^i \sigma^i(\Lambda F) \cdots \sigma^7(F) \\ &= \sum_{i=0}^{7} \sigma^0(F) \cdots \varepsilon^i \sigma^i(\Lambda) \sigma^i(F) \cdots \sigma^7(F) \\ &= (\sum_{i=0}^{7} \varepsilon^i \sigma^i(\Lambda)) \prod_{i=0}^{7} \sigma^i(F) = \bar{\Lambda}\bar{F}. \end{aligned}$$

Thus, $\bar{F}$ is a Darboux polynomial of d with eigenvalue $\bar{\Lambda}$. Since ε is a primitive eighth root of 1, we have

$$\sum_{i=0}^{7} \varepsilon^{ri} = \frac{1-\varepsilon^{8r}}{1-\varepsilon^r} = 0, \text{for any} \;\; r < 8.$$

Thus

$$\begin{aligned}
\bar{\Lambda} &= \sum_{i=0}^{7} \varepsilon^i \sigma^i(\Lambda) \\
&= \sum_{i=0}^{7} \varepsilon^i \sigma^i(k_1 x + k_2 y + k_3 z + k_4 t) \\
&= \sum_{i=0}^{7} \varepsilon^i (k_1 \varepsilon^{3i} x + k_2 \varepsilon^{5i} y + k_3 \varepsilon^{3i} z + k_4 \varepsilon^i t) \\
&= k_1 \sum_{i=0}^{7} \varepsilon^{4i} x + k_2 \sum_{i=0}^{7} \varepsilon^{6i} y + k_3 \sum_{i=0}^{7} \varepsilon^{4i} z + k_4 \sum_{i=0}^{7} \varepsilon^{2i} t \\
&= 0.
\end{aligned}$$

Therefore, $D(\bar{F}) = 0$. It is a contradiction to the fact that $k(X)^d$. Hence, d has no Darboux polynomials. $\square$

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